Formative Assessment Lesson Materials for

Summing Consecutive Numbers

Mathematical goals

This lesson unit is intended to help you assess how well students are able to:

- Make and investigate mathematical conjectures
- Explore properties of numbers
- Express mathematical arguments

Common Core State Standards

This lesson involves a range of mathematical practices from the standards, with emphasis on:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

This lesson asks students to select and apply mathematical content from across the grades, including the content standards:

- A.SSE – Seeing Structure in Expressions
- A.CED – Creating Equations

Introduction

In this unit, students make and investigate conjectures to solve a mathematical problem.

- Before the lesson, students attempt the task *Summing Consecutive Numbers* individually. You then review their work, and create questions for students to answer in order to improve their solutions.
    - Questions created may be in response to individual student work, or collective need.
- At the start of the lesson portion, students again work individually on *Summing Consecutive Numbers*, answering your questions.
- Next, students work collaboratively in small groups. Their task is to produce a better solution to *Summing Consecutive Numbers* than they did individually. Then, working in the same small groups, students analyze *Sample Student Responses to Summing Consecutive Numbers*.
- In a whole-class discussion, students compare and evaluate the mathematical arguments they have seen and used.

NCTM's Problem to Ponder for Aug 17, 2010 and www.nrich.maths.org were heavily consulted in writing this FAL.

- In the final part of the lesson, students spend ten minutes reviewing their individual solutions, and writing about what they have learned or improving their initial solution.

Materials required

- Each individual student will need a copy of the task sheet *Summing Consecutive Numbers.*
- Each small group of students will need a new copy of the task sheet *Summing Consecutive Numbers*, a sheet of poster paper, and a copy of the *Sample Student Responses*.
- The Excel file *SumofConsecutive* for those students working with the extension problem.

Time needed

Approximately fifteen minutes before the lesson and a one-hour lesson. Timings given are only approximate. Exact timings will depend on the needs of your class.

Before the lesson

Assessment task: Summing Consecutive Numbers (15 minutes)

Have the students do this task, in class or for homework, a day or more before the lesson. This will give you an opportunity to assess their work, and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Summing Consecutive Numbers*. Introduce the task briefly. Help the class to understand the problem.

It is important that students answer the questions without assistance, as far as possible. If students are struggling to get started, ask questions that help them understand what they are being asked to do, but do not do the problem for them. See the *Common Issues* table on the following page.

Students should not worry too much if they cannot understand or do everything, because soon there will be a lesson using the same task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding, and their problem solving strategies.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare scores, and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given the *Common Issues* table on the next page. We suggest that you make a list of your own questions, based on your students' work, using the ideas on the following page. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

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Common issues:	Suggested questions and prompts:
Students have a difficult time getting started	• *Can you give an example of a set of two or more consecutive numbers and the sum of those numbers?* • *Have you determined if any of the numbers 1 – 30 can be written as the sum of consecutive numbers?*
Student chooses different consecutive sums, but the selections don't appear to be systematic.	• *Can you organize your work in a way that would make sense to others?* • *Can you describe your method for selecting the sets of consecutive numbers you're summing?*
Student systematically tries various sets of consecutive numbers, but isn't making a connection to any "hidden structures".	• *What do you notice about the sums of two consecutive numbers? Three? Four?* • *What do you notice about the numbers you can't make?* • *How can you express the sum of a set of numbers algebraically? Can you algebraically express the sum of any set of consecutive numbers?* • *Have you tried representing these numbers in a non-algebraic way, in order to gain a new perspective on the relationship?*
For students who attempt to express the pattern algebraically, but are having difficulty.	• *Have you tried representing these numbers in a non-algebraic way, in order to gain a new perspective of the relationship?* • *Is there another way to represent each number in a set algebraically?* (Students may write a three number set as x, $x+1$, $x+2$ and not realize that the numbers can be written $x-1$, x, $x+1$) • *What does x represent in your expression/ equation? Does x have to be the first number?*

Students find the initial task/ lesson task too easy.	• Invite these students to explore the extension activity, which asks them to determine "how many different ways can a given whole number be written as a consecutive sum"?
	    ○ *How can you systematically organize your work? Have you investigated the numbers 1-30?*
	    ○ *What was your earlier conjecture, and what conclusion did you draw while investigating that conjecture?*
	    ○ *What is the significance of each factor in the prime factorization of a consecutively summed number?*
	    ○ *What conjectures have you posed to further explore this problem?*

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Suggested lesson outline

Introduction and individual work (10 minutes)

Return students their solutions to *Summing Consecutive Numbers*. If you have chosen not to write questions on individual student papers, display your list of questions on the board.

[Last lesson] you worked on writing conjectures about properties of numbers and constructing mathematical arguments to prove your conjectures. Do you recall what the task was about? I have read your solutions, and I have some questions I'd like you to think about. Work individually for ten minutes, answering my questions to improve your work.

Collaborative small-group work (20 minutes)

Organize students into groups of two or three. Give each group a new copy of the task sheet *Summing Consecutive Numbers*, and a large sheet of poster paper.

I'd like you to put your solutions to one side now.

Start afresh on the same pattern problem. I want you to work together in your groups, to produce a better solution together than you each did individually.

I'd like your group to decide on one key conclusion you've reasoned through and to share your conclusion on a poster. Make sure you write down all your reasoning, and label everything clearly.

To begin, I'd like you to take turns in your groups to share your conjectures and arguments. Think about when you were working alone. What properties did your group mates notice that you didn't? Would their observations help you notice a pattern?

You have two roles while students are working, to find out about student methods, and to support student problem solving.

Find out about student methods

As students continue to make and investigate their conjectures, circulate about the room; the purpose of circulating is to listen and monitor student methods/ arguments. Use the information you gleaned from the initial, individual task to decide where to focus your attention.

Research indicates that teachers who can have thoughtfully considered anticipated student responses are better able to address those misconceptions if they occur; moreover, these teachers are better able to address student misconceptions that were not anticipated when/ if they arise.

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- Try to determine if students are making progress beyond their initial work, and if they aren't remind them of the feedback you have provided to them, either via whole class feedback or individual feedback.
- Are students able to reconcile the approach(es) of their partners to their own work, if the partner's work is different?
- Are some students jumping into relatively more complicated examples to explore, and forgetting the role simpler cases/ analogies can play in problem solving?
- Are students organizing their work in a systematic way, keeping track of insights as they arise?
- Are students exploring different representations, such as geometric or algebraic representations, if they are getting stuck in repeated numeric trials?
- Do students realize the importance of a "median" number? Odd factor?
- Are students able to explain how the identification of an odd factor can lead them to a set of consecutive terms, summing to a number in question?
- Once students arrive at a comfortable solution, do they reflect on their approach and communicate the approach, conclusion, and reasoning in a mathematically precise way?

Support student problem solving

Try not to prompt students into using a particular problem solving method, and try not to point out the difficulties with their chosen methods to them. Instead, ask questions to prompt students to justify, and evaluate their own solution strategies.

The questions in the *Common Issues* table might guide your questioning, as might the questions in the preceding section.

Prompt students to share the reasoning behind their conjectures so that other groups can understand what is written.

If any group finishes their solution, or if a group finds the initial task too easy, ask them the extension question, "How many *different* ways can a given whole number be written as a consecutive sum?"

Collaborative analysis of *Sample Student Responses* (10 minutes)

Give a copy of the three *Sample Student Responses* to each small group of students. Ask students to read the solutions, and to answer the questions together.

Looking at your posters, I can see you have used a range of different methods to solve this problem. I'm giving you some work produced by students from another class on the same problem. I would like you to answer these questions about each student's work:

- *Read through the solution and make sense of how the student is solving this problem.*
- *Figure out what conjectures the student made, and see if they attempted to construct a viable argument to support their conjecture. What evidence do you see of a viable argument?*
- *Are the arguments based on a few examples, lots of examples, mathematical properties?*
- *Is the argument clear and precise, or does it leave you wondering what the student-author meant?*
- *Decide what is good about the work, and what feedback you might provide to the student-author to improve their work.*

During small-group work, support students as they work. If students find it difficult to get started, suggest one group mate read the solution aloud, slowly, as others follow along. It may be useful to assign another group mate the role of making sure that everyone in the group understands the argument; this can be done by asking the reader to pause intermittently so that understanding can be checked across the group.

[To the reader] Stop reading intermittently, so that understanding can be checked across the group.

[To the group] Explain what the conjecture is and how the student is arguing support for their conjecture. It may be useful to distinguish arguments based on the representation the student used, such as algebraic or geometric, and whether the argument is based on examples or properties.

Plenary whole-class discussion: comparing different approaches (10 minutes)

Organize a whole-class discussion of *the Summing Consecutive Numbers*. Focus the discussion on the methods students have seen and used during the lesson, rather than discussing who has the "best" argument or a "correct" solution.

In particular, ask students to discuss the strengths and weaknesses of the different arguments they have seen or heard in the *Sample Student Responses:*

JoLisa

- Noted that "all odd numbers, except one, can be written as the sum of consecutive numbers" and used an algebraic method to argue her point, although she needs to define the variables
- Said it appears that "multiples of 3 can be written as consecutive sum" but there is not argument made to support this conjecture.
- Also believes that "multiples of ten" can be written, but bases her argument on what appears to be only four examples.

Alydia

- Also notes that "all odd numbers can be written as consecutive sums" and argues her point using algebra, but still needs to define what n represents. She may need to go on to explain why $2n + 1$ would be odd, if this is a new idea for the class.
- Alydia also observes that multiples of three can be expressed as the sum of three consecutive numbers, and argues her point using algebra, but still needs to refine this argument for the same reasons stated above.
- Alydia goes on to offer eight more conjectures, although the conjectures aren't always clearly stated, and argues each point using the same logic as in the previous two examples; that is, she uses algebra, sums a set of numbers, and adds the next consecutive number, then uses properties of number and algebra to draw a conclusion.

Josh

- Josh uses a geometric model to represent the consecutive sums. His first conjecture that "all odd #'s can be written as a sum of 2 consecutive whole #'s is incomplete, and is based on 3 examples.
- He also observes that multiples of three can be expressed as the sum of three consecutive numbers, and argues his point using a "geometric pattern", but does not describe the "pattern" and needs to refine this argument.
- Josh loses an ability to make a clear generalization regarding a tower with a base of four (sum of four consecutive numbers) and moves to gathering previous data to see if a conjecture can be made.
- It appears that Josh has reasoned that "numbers with at least one odd factor can be written as a sum of consecutive numbers" but it is unclear how he

arrived at this conclusion. He also makes no clear, formal argument as to why this is true.

What are some of the qualities of a precise and mathematical argument?

What are some of the qualities of a viable mathematical argument? In what ways did the viability vary across the work we saw today?

In what ways could you improve JoLisa, Alydia, or Josh's arguments?

How did seeing different arguments and conjectures help you refine your own thinking on this problem?

Ask students to contribute, with reference to their own posters. Try to avoid making evaluative comments yourself. Instead, encourage students to respond to other students' explanations

Try to help students understand that investigating conjectures when making sense of mathematical problems can shed light on the original question, even if none of the investigated conjectures fully address the original question.

If some of your students were given the *Extension Question* (How many different ways can a given whole number be written as a consecutive sum?) save their sharing for last. Try to focus the class' attention on the conjectures and method of argument when *Extension* groups share out.

Review individual solutions to the assessment task (10 minutes)

Ask students to reread their original solutions, and write about what they have learned during the lesson.

Read through your original solution.

- *What have you learned during the lesson?*
- *Suppose you have to work on a new pattern finding or conjecture problem. What advice would you give yourself?*
- *Revise your original work to either make it better or make a connection to another method you saw today.*

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Which Whole Numbers Can Be Written as the Sum of Consecutive Whole Numbers?

Consider, for example, that 6 = 1 + 2 + 3 and 15 = 4 + 5 + 6. Thus, 6 and 15 are whole numbers that can be written as consecutive sums. It is also the case that 15 = 7 + 8.

Try to answer the question above, and if along your solution path you see something interesting, pose another question and try to answer it too. Support your conclusions with convincing arguments or proofs.

Extension Question

How many different ways can a given whole number be written as a consecutive sum?

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Conjectures and arguments to *Summing Consecutive Numbers*

Any number with at least one odd factor can be written as the sum of consecutive terms

Let N=(2n+1)*k, where N is the consecutive sum and (2n+1) is an odd whole number, n>0 and k is a whole number. By the distributive and commutative properties N = (2n+1)k = ((1+1)n+1)k = (n+n+1)k = (n+1+n)k = nk+k+nk. Thus, if an odd factor of N is known, then a consecutive set of number may be determined because k can also be determined.

For example:

Let N = 30 = 5*6. 5 is an odd factor and we know the "factor pair" given 5, so we can let k = 6 in this example.

6
Median

5	6	7
(6-1)	6	(6+1)

4	5	6	7	8
(6-2)	(6-1)	6	(6+1)	(6+2)

If we instead chose the factor pair 15*2=30, then k=2 and there would be seven terms on either side of 2 (do you see why?). In the example of 15*2, it is also the case the final "simplified" set of terms won't be 15.

-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

But we aren't allowing our consecutive numbers to be negative! No worries, because we can combine additive inverses the above set of terms can be rewritten as

6 7 8 9

In this line of argument though, since we "temporarily" allow negative numbers, how can we be guaranteed that the "simplified" set will only contain whole numbers? The sums of consecutive whole numbers must be positive, so N=(2n+1)k must also be positive. 2n+1 is the positive "count" of k's in the set of consecutive numbers, so k must also be positive, meaning the smallest k can be is 1. So there will always at least be two more positive terms than negative terms in the "unsimplified" set because

-1 0 1 2 3

The beginnings of a geometric argument, from the NCTM President's Aug 17, 2010, found at http://nctm.org/about/content.aspx?id=26376

Solution:

The August Problem to Ponder was this: Which whole numbers can be written as the sum of consecutive whole numbers? An extension also asked, How many different ways can a given whole number be written as the sum of consecutive whole numbers?

One of the first observations that many people make on this problem is that every odd number can be written as the sum of *two* consecutive whole numbers, for example,

5 = 2 + 3, 15 = 7 + 8, and 87 = 43 + 44. In general, since every odd whole number can be written in the form $2n + 1$ where n is some other whole number, and since $(2n + 1) = n + (n + 1)$, then every odd whole number can be expressed as the sum of two consecutive whole numbers.

You may also have discovered that some whole numbers cannot be written as any consecutive sum. For example, 2 cannot be written as the consecutive sum of whole numbers, because the only possible choices for addends are 0 and 1. Another number that cannot be written as a consecutive sum is 8, because 2 + 3 + 4 = 9 (already too big) and 1 + 2 + 3 = 6 (too small). Because all other possibilities are much bigger or much smaller, we can prove that 8 cannot be written as a consecutive sum of whole numbers.

So far, we now know that all odd numbers *can* be written as a consecutive sum, that some even numbers can be, but that other even numbers cannot. We also know that many numbers can be written as a consecutive sum in more than one way.

Which even numbers cannot be written as a consecutive sum? Why? And, for those numbers that can be written as a consecutive sum, how can we predict the number of different ways that they can be written as a consecutive sum?

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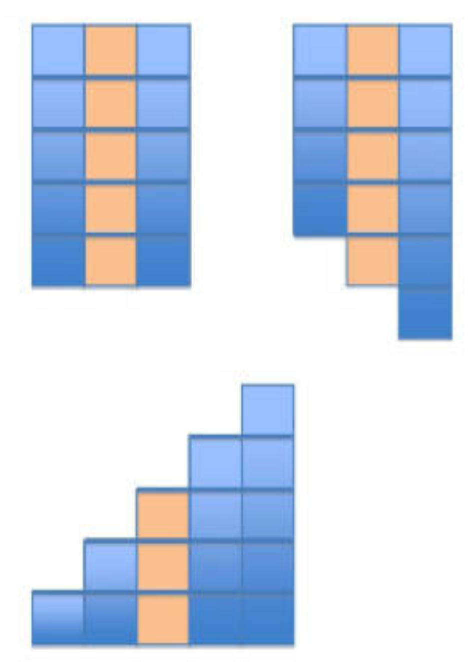
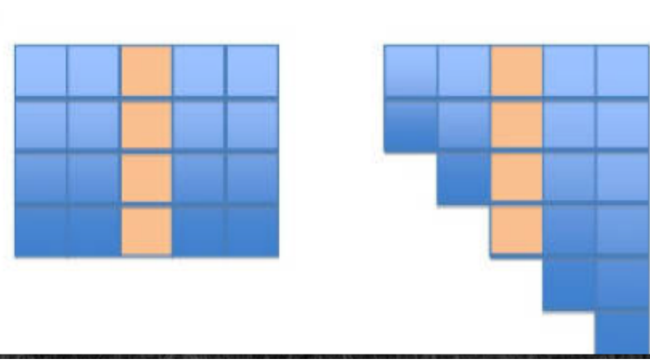
15 = 3 x 5 can be represented by a rectangular grid (see left figure below). The area of this figure can then be rearranged into columns that represent the consecutive sum 4 + 5 + 6, as seen in the figure on the right below. Similarly, 15 = 5 x 3 can be rearranged into another consecutive sum, 1 + 2 + 3 + 4 + 5 as shown in the third figure below. In each case, we can make "trades" of the squares in the grid across the middle column (shown in a different color). The height of the middle column is one of the factors of 15 (5 in the first case, and 3 in the second). As long as we have an *odd* factor in our number, we can make these trades, and write our number as a consecutive sum.



Similarly, we can represent 20 = 5 x 4 in a rectangular grid if we create 5 copies of 4, and we can trade cubes across the middle column to write the consecutive sum 20 = 2 + 3 + 4 + 5 + 6.

In this way, we can create a consecutive sum determined by each odd factor of a number. Any number with an odd factor can be written as a consecutive sum. So, what numbers cannot be written as consecutive sums? You should have the answer by now!



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Rigorous proof by some British students, from http://nrich.maths.org/507/solution

Jack, Paul and Matthew also sent in a correct solution, and again, the reasoning below may help to explain why it works.

Like the number 15, all odd numbers can be written as the sum of two consecutive integers. For any odd number k, we know that $k-1$ and $k+1$ are even, also $(k-1)/2$ and $(k+1)/2$ are consecutive integers, so we can write $k=(k-1)/2+(k+1)/2$.

All numbers of the form $N=pk$ where p and k are integers, and k is odd and greater than one, can be written as the sum of consecutive integers.

As before we use the integers $(k-1)/2$ and $(k+1)/2$ but this time we have to add $2p$ consecutive integers, p of them less than $(k/2)$ and p of them greater than $(k/2)$, so that their mean is $k/2$ and their sum is $(2p)\times(k/2)=pk$

For example $N=44=4\times 11$ can be written as the sum of 8 consecutive integers, 4 less than 5.5 and 4 greater than 5.5 so that the sum is $8\times 5.5=44$.

$$\begin{aligned}44 &= (11-7)/2+(11-5)/2+(11-3)/2+(11-1)/2+(11+1)/2+(11+3)/2+(11+5)/2+(11+7)/2\\ &= 2+3+4+5+6+7+8+9\end{aligned}$$

This construction works for all numbers of the form $N=pk$ where p and k are integers and k is odd and greater than one, but the consecutive sum may include some negative integers. If this happens then a string of terms reduces to zero giving fewer terms in the final expression.
By this method,

$$\begin{aligned}12 &= 4\times 3\\ &= (-2)+(-1)+0+1+2+3+4+5\\ &= 3+4+5\end{aligned}$$

giving three terms in the final expression rather than eight.

If a number is a power of 2 then it is impossible to write it as a sum of consecutive integers. Consider the sum of consecutive integers starting at $(q+1)$ and ending at p . There are $(p-q)$ terms in this sum. The mean of the terms is $(p+q+1)/2$ so we can write:

$$(q+1)+(q+2)+\ldots+(p-1)+p=(p-q)(p+q+1)/2$$

As $(p-q)$ and $(p+q+1)$ cannot both be even, this means that the sum of consecutive integers must represent a number which has at least one odd factor (other than the factor 1) so a power of 2 cannot be written as a sum of consecutive integers.

We use the fact here that if $(p-q)$ is odd then $(p+q)$ is odd and $(p+q+1)$ is even, or if $(p-q)$ is even then $(p+q)$ is even and $(p+q+1)$ is odd.

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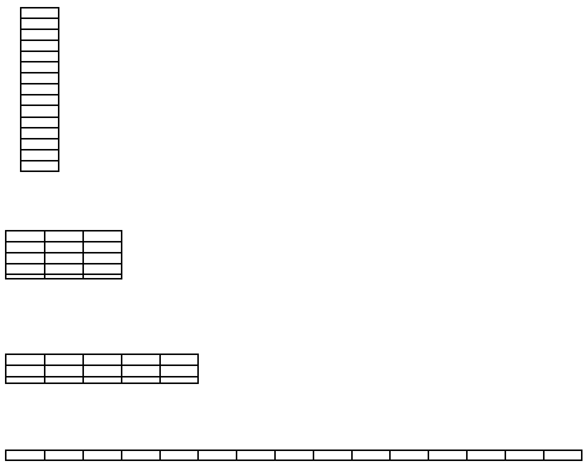
Conjectures and arguments to *Extension Question: How many different ways can a given whole number be written as consecutive sum?*

Drawing on arguments from above, we know that the odd factors in the prime factorization of a given number determine whether a whole number may be written as a sum of consecutive numbers. We also define a "consecutive sum" as a set of terms that has more than one term.
Let's look at an example, then we will share the answer.

If the given whole number is 15, then it can be represented in the following ways:



b*h = (2*int(b/2) +1) * h

1*15 = (2*0 + 1)*15

3*5 = (2*1 + 1)*5

5*3 = (2*2+1)*3

15*1 = (2*7+1)*1

This indicates that 15 can be written in three different ways, because the top rectangle and factor pair show that there are 0 terms (int(1/2) = 0) to either side of the median term (15) meaning this is not a "consecutive sum".

The four possible arrangements of terms exist because of the prime factorization of 15.

$15 = 3^1 * 5^1$

$15 = (1) * (15) = (3^0 * 5^0) * (3^1 * 5^1)$

$15 = (3) * (5) = (3^1 * 5^0) * (3^0 * 5^1)$

$15 = (5) * (3) = (3^0 * 5^1) * (3^1 * 5^0)$

$15 = (15) * (1) = (3^1 * 5^1) * (3^0 * 5^0)$

So, for this example, $3^a * 5^b$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$, thus there are 2*2 ways to write the first factor in each factor pair, but we may not count the first factor pair, 1*15. The second factor depends entirely on the first, so we only need to count the number of ways to write the first factor in the factor pair.

Let's look at another example, where the given number is 36. Without referencing the geometric representation, we can write the factor pairs of 36 as:

$36 = (1) * (36) = (2^0 * 3^0) * (2^2 * 3^2)$

$36 = (2) * (18) = (2^1 * 3^0) * (2^1 * 3^2)$

$36 = (3) * (12) = (2^0 * 3^1) * (2^2 * 3^1)$

$36 = (4) * (9) = (2^2 * 3^0) * (2^0 * 3^2)$

$36 = (6) * (6) = (2^1 * 3^1) * (2^1 * 3^1)$

$36 = (9) * (4) = (2^0 * 3^2) * (2^2 * 3^0)$

$36 = (12) * (3) = (2^2 * 3^1) * (2^0 * 3^1)$

$36 = (18) * (2) = (2^1 * 3^2) * (2^1 * 3^0)$

$36 = (36) * (1) = (2^2 * 3^2) * (2^0 * 3^0)$

Many of these equations aren't useful to us because the "base" of our rectangle must by odd, so we can only consider three of these equations, those that are bold. Moreover, we must also remove the first bold termed equation for reasons mentioned above.
For this example $2^a * 3^b$, where $0 \leq a \leq 2$ and $0 \leq b \leq 2$, thus there are 3 ways to write the first factor in each factor pair *if we want the first factor to be odd*; we do this by ignoring the 2 in the prime factorization when counting the possibilities. As stated above, we must remove one of the 3 ways, due to our definition of "consecutive sum".

As these two examples suggest, to find the number of different ways a given whole number may be written as a consecutive sum an algorithm may be followed:

1. Find the prime factorization of the given number.
2. Add one to each exponent of the odd factors in the prime factorizations, and then multiply all the "exponents plus one" together.
3. Subtract one from this product of exponents, to account for the factor pair 1*N = N

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